

PROJECTIVELY UNIVERSAL COUNTABLE METRIZABLE GROUPS

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ABSTRACT. We prove that there exists a countable metrizable topological group G such that every countable metrizable group is isomorphic to a quotient of G . The completion H of G is a Polish group such that every Polish group is isomorphic to a quotient of H .

1. INTRODUCTION

A topological group is *Polish* if it is homeomorphic to a complete separable metric space. It has been known for about 30 years that there exist *injectively universal* Polish groups [13, 14], that is, Polish groups G such that every Polish group H is isomorphic (as a topological group) to a (necessarily closed) subgroup of G . Those examples in particular answered a Scottish Book question by Schreier-Ulam (question 103 in [8]); for a recent new example, see [3]. A few years ago L. Ding proved [6], answering a long-standing question of A. Kechris (see [7, Problem 2.10], [2, Problem 1.4.2]), that there also exists a *projectively universal*, or *couniversal*, Polish group, that is, such a Polish group G that every Polish group H is isomorphic (as a topological group) to the quotient group G/N for some closed invariant subgroup $N \triangleleft G$.

The aim of this note is to provide a shorter proof of a stronger theorem: there exists a projectively universal countable metrizable group. The completion of such a group is a projectively universal Polish group (Theorem 2.1), so our result indeed implies that of Ding. We give two constructions in sections 3 and 4, due to the first and the second author, respectively.

We mention that a projectively universal Abelian Polish group was constructed in [11], and an injectively universal Abelian Polish group was constructed in [12]. The question remains open, due to Kechris,

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whether every Polish group is isomorphic to a quotient of a subgroup of the unitary group $U(\ell^2)$; the answer is positive in the Abelian case [5], [15].

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2. POLISH GROUPS AS COMPLETIONS OF COUNTABLE GROUPS

We are going to explain why the completion of a projectively universal countable metrizable group is a projectively universal Polish group. All our groups are Hausdorff. Recall that for every topological group G its *Rajkov completion* \hat{G} is defined as the completion with respect to the upper bound of the left and right uniformities, see [10, 1]. If G is a topological subgroup of the group $\text{Iso}(X)$ of linear isometries of a Banach space X (every topological group admits such an embedding), then we can take for \hat{G} the closure of G in $\text{Iso}(X)$. If G is a (necessarily metrizable) group with a countable base, then G is Polish if and only if G is Rajkov complete, that is, $G = \hat{G}$. Indeed, if G is Rajkov complete, then the two-sided uniformity on G is complete and metrizable, hence admits a compatible complete metric. It follows that G is Polish. Conversely, suppose that G is Polish. Every Polish space X is a G_δ subset in any Hausdorff space $Y \supset X$ containing X as a dense subset. If $G \neq \hat{G}$, then G is a dense G_δ subset in the Polish group \hat{G} , and so is the translate Gx for any $x \in \hat{G} \setminus G$. Since $G \cap Gx = \emptyset$, we obtain a contradiction with the Baire Category Theorem.

Thus every Polish group can be viewed as the Rajkov completion of any of its dense countable subgroups. Recall that the quotient of any Polish group by a closed invariant subgroup is Polish [2, Proposition 1.2.3], [1, Theorem 4.3.26]. Theorem 4.3.26 in [1] actually is more general and deals with Čech-complete groups, but for spaces with a countable base ‘Polish’ and ‘Čech-complete’ are equivalent.

Theorem 2.1. *Let G be a projectively universal countable metrizable group. Then the completion \hat{G} is a projectively universal Polish group.*

Proof. Write the given Polish group as the completion \hat{H} of a countable metrizable group H . Write H as a quotient of G . Then $H = G/N$ can be identified with a dense subgroup of \hat{G}/\hat{N} [4, Ch. 3, §2, Proposition 21]. The quotient \hat{G}/\hat{N} of a Polish group \hat{G} is Polish, hence Rajkov complete. It follows that \hat{H} can be identified with \hat{G}/\hat{N} , a quotient of \hat{G} . \square

3. THE FIRST CONSTRUCTION

3.1. We will construct a couniversal countable metrizable group using a technique developed by Roelcke and Dierolf [10].

For a sequence $(B_n)_{n=1}^\infty$ of subsets of a group G , define their symmetric product as follows:

$$[B_n] = [B_n]_{n=1}^\infty = \bigcup_{n=1}^\infty \bigcup_{\sigma \in S_n} B_{\sigma(1)} \cdot B_{\sigma(2)} \cdot \dots \cdot B_{\sigma(n)}.$$

For example, the proof of the Birkhoff–Kakutani theorem implies:

Lemma 3.1. *If $(V_n)_{n=1}^\infty$ be a sequence of neighbourhoods of the identity in a topological group G such that $V_n^{-1} = V_n$ and $V_{n+1}^2 \subseteq V_n$ for all n . Then for every k one has*

$$[V_n]_{n=k+2}^\infty \subseteq V_k.$$

Let now \mathcal{F} be a filter of subsets of a group G . For a mapping $\Phi: G \rightarrow \mathcal{F}$ denote

$$\mathcal{V}_\Phi = \bigcup_{g \in G} g^{-1}(\Phi(g) \cup \Phi(g)^{-1})g.$$

According to [10], the sets of the form $[\mathcal{V}_{\Phi_n}]_{n=1}^\infty$, where (Φ_n) runs over all sequences of maps from G to \mathcal{F} , form a neighbourhood basis at identity in the finest group topology on G in which $\mathcal{F} \rightarrow e$. E.g., in [9] this was used to describe a neighbourhood basis in the free topological group on a uniform space.

Since it is obvious that every set of the form $[\mathcal{V}_{\Phi_n}]$ contains an element of the filter \mathcal{F} , the result of Roelcke and Dierolf follows from Lemma 3.1, as well as the following easily verifiable results.

Lemma 3.2. *Every set $[\mathcal{V}_{\Phi_n}]$ is symmetric.*

Lemma 3.3. *Assuming the sequence of mappings $\Phi_n: G \rightarrow \mathcal{F}$ is pointwise monotone (that is, $\Phi_{n+1}(g) \subseteq \Phi_n(g)$ for each $g \in G$), we have*

$$[\mathcal{V}_{\Phi_{2n}}]^2 \subseteq [\mathcal{V}_{\Phi_n}].$$

Lemma 3.4. *For each $h \in G$, denote Φ^h the right translate of Φ by h , that is, $\Phi^h(g) = \Phi(gh)$. Then*

$$h^{-1}[\mathcal{V}_{\Phi_n^h}]h \subseteq [\mathcal{V}_{\Phi_n}].$$

Corollary 3.5. *Let $\Phi_n: G \rightarrow \mathcal{F}$ be a pointwise monotone sequence of mappings. Then the sets*

$$[\mathcal{V}_{\Phi_{kn}^h}], \quad h \in G, \quad k \in \mathbb{N},$$

form a subbasis at identity for a group topology on G .

3.2. On the set \mathbb{N} of natural numbers select a decreasing sequence of sets (U_n) with empty intersection and such that $U_0 = \mathbb{N}$ and $U_n \setminus U_{n+1}$ is infinite for each n .

Let $F(\mathbb{N})$ denote the free group on the set \mathbb{N} of free generators. Define a sequence of functions $\phi_n: F(\mathbb{N}) \rightarrow \mathbb{N}$ by setting $\phi_n(k^{\pm 1}) = n + k$ for all $k \in \mathbb{N}$ and extending each ϕ_n over the free group by recursion on the length of a reduced word $w = n_1^{\varepsilon_1} \dots n_k^{\varepsilon_k}$, $\varepsilon_i = \pm 1$, $n_i \in \mathbb{N}$ as follows:

$$\phi_n(w) \equiv \phi_n(n_1^{\varepsilon_1} \dots n_k^{\varepsilon_k}) = \max \left\{ \phi_{\phi_n(n_1^{\varepsilon_1} \dots n_{k-1}^{\varepsilon_{k-1}})}(n_k), \phi_{\phi_n(n_2^{\varepsilon_2} \dots n_k^{\varepsilon_k})}(n_1) \right\}.$$

Now set $\Phi_n(g) = U_{\phi_n(g)}$. This is easily seen to be a pointwise monotone family of maps from the free group to a filter generated by (U_n) .

As in Corollary 3.5, the family (Φ_n) defines a metrizable group topology on $F(\mathbb{N})$. This topology is Hausdorff: for every k , the neighbourhood $[\mathcal{V}_{\Phi_{kn}}]$ is contained in the normal subgroup generated by U_k , and such subgroups separate points in the free group.

We will now show that when equipped with the above topology, $F(\mathbb{N})$ is a couniversal countable metrizable group.

3.3. Let (V_n) be a countable basis of neighbourhoods of identity of a metrizable group G . For every element g , define the *scale* of g with regard to the fixed basis as a function $\theta_g: \mathbb{N} \rightarrow \mathbb{N}$, as follows:

$$\theta_g(n) = \min\{m: g^{-1}V_m g \cup gV_m g^{-1} \subseteq V_n\}.$$

For a general Polish group such as $\text{Homeo}_+[0, 1]$ for instance, it is easy to see that the scales formed with regard to any neighbourhood basis form a cofinal subset of $\mathbb{N}^{\mathbb{N}}$. Not so for countable metrizable groups.

Lemma 3.6. *A countable metrizable group G admits a symmetric neighbourhood basis (V_n) of identity satisfying $V_{n+1}^2 \subseteq V_n$ for all n and with regard to which the scale of every element satisfies $\theta_g(n) \leq n + m$ for a suitable $m = m(g)$.*

Proof. Enumerate $G = \{g_m: m \in \mathbb{N}_+\}$ and choose a basis V_n recursively so that for each n , $g_m^{-1}V_{n+1}g_m \cup g_mV_{n+1}g_m^{-1} \subseteq V_n$ whenever $m \leq n$. \square

Lemma 3.7. *Let $U \subseteq \mathbb{N}$ be infinite, let V be a countably infinite set, and let $m: V \rightarrow \mathbb{N}$ be a function whose image contains 0. There exists a surjection $f: U \rightarrow V$ such that*

$$\forall k \in U, \quad m(f(k)) \leq k.$$

Proof. Identify V with \mathbb{N} in such a way that $m(0) = 0$. Define $f(k)$ recursively in k as the smallest element of $\mathbb{N} = V$ not yet chosen provided its image under m does not exceed k , and 0 otherwise. \square

Now choose a basis $(V_n)_{n=0}^\infty$ for G as in Lemma 3.6, where $V_0 = G$. The corresponding function m on G satisfies $m(e) = 0$. For every $n \in \mathbb{N}$ apply Lemma 3.7 to the sets $U = U_n \setminus U_{n+1}$, $V = V_n$, and the function m . We obtain a surjection $f_n: U_n \setminus U_{n+1} \rightarrow V_n$ with $m(f_n(k)) \leq k$ for all $k \in U_n \setminus U_{n+1} \subseteq \mathbb{N}$. Amalgamate all f_n to obtain a surjection $f: \mathbb{N} \rightarrow G$. For all $k, n \in \mathbb{N}$ one has

$$(3.1) \quad \phi_n(k) = n + k \geq n + m(f(k)) \geq \theta_{f(k)}(n).$$

Extend f to a surjective group homomorphism $\bar{f}: F(\mathbb{N}) \rightarrow G$.

Lemma 3.8. *For all $x \in F(\mathbb{N})$ and $n \in \mathbb{N}$, one has $\theta_{\bar{f}(x)}(n) \leq \phi_n(x)$.*

Proof. Induction on the reduced length, $\ell(x)$, of x . For $\ell(x) = 1$, this is Eq. (3.1) and the symmetry of θ . Suppose the result holds for $\ell(x) \leq \ell$. Let $x = x_1^{\varepsilon_1} \dots x_{\ell+1}^{\varepsilon_{\ell+1}}$, where $\varepsilon_i = \pm 1$, be an irreducible word. We have:

$$\begin{aligned} f(x)^{-1} V_{\phi_n(x)} f(x) &\subseteq \bar{f}(x_2^{\varepsilon_2} \dots x_{\ell+1}^{\varepsilon_{\ell+1}})^{-1} V_{\phi_n(x_2^{\varepsilon_2} \dots x_{\ell+1}^{\varepsilon_{\ell+1}})} \bar{f}(x_2^{\varepsilon_2} \dots x_{\ell+1}^{\varepsilon_{\ell+1}}) \\ \text{induction hypothesis} &\subseteq \bar{f}(x_2^{\varepsilon_2} \dots x_{\ell+1}^{\varepsilon_{\ell+1}})^{-1} V_{\theta_{\bar{f}(x_2^{\varepsilon_2} \dots x_{\ell+1}^{\varepsilon_{\ell+1}})}(n)} \bar{f}(x_2^{\varepsilon_2} \dots x_{\ell+1}^{\varepsilon_{\ell+1}}) \\ &\subseteq V_n. \end{aligned}$$

Similarly, $f(x) V_{\phi_n(x)} f(x)^{-1} \subseteq V_n$, and we conclude. \square

As an application of the lemma, $\bar{f}(\mathcal{V}_{\Phi_n}) \subseteq V_n$ for every $n \in \mathbb{N}$. Now Lemma 3.1 implies that for every k , $\bar{f}(\bigcup_{n=k+2}^\infty \mathcal{V}_{\Phi_n}) \subseteq V_k$. Consequently, the homomorphism \bar{f} is continuous. Since each neighbourhood of identity in $F(\mathbb{N})$ contains one of the sets U_n which is being mapped by \bar{f} onto V_n , the homomorphism \bar{f} is also open.

4. THE SECOND CONSTRUCTION

Another construction of a projectively universal countable metrizable group is based on the following idea: take a collection of size 2^ω representing, up to an isomorphism, all possible countable metrizable groups. The product of this collection contains a countable dense subgroup that admits an open projection onto any factor. Such a group is not metrizable, but it is possible to refine its topology so that one gets a metrizable group while all projections onto factors remain open.

Let G be an injectively universal topological group with a countable base. For example, we can take for G the group $\text{Iso}(U)$ of isometries of the Urysohn space or the group $H(Q)$ of all self-homeomorphisms of the Hilbert cube. Consider the countable power $G^\mathbb{N}$ of G , and let X be

$G^{\mathbb{N}}$ equipped with a finer zero-dimensional topology with a countable base \mathcal{B} . We assume that \mathcal{B} consists of clopen sets and is closed under complements, finite unions and hence also under finite intersections. In other words, \mathcal{B} is a Boolean algebra of clopen sets.

We consider each $x \in X = G^{\mathbb{N}}$ as an index of a certain countable metrizable group G_x , namely, the subgroup of G generated by the elements of the sequence x . Every countable metrizable group is of the form G_x for some $x \in X$. Consider the group (without topology) $C(X, G)$ of all continuous maps $f : X \rightarrow G$. We are going to construct a countable subgroup $K \subset C(X, G)$ and a metrizable group topology on K such that for every $x \in X$ the evaluation map $f \mapsto f(x)$ from K to G is an open map onto G_x . This implies that K is a projectively universal countable metrizable group.

Let $p_n \in C(X, G)$ be defined by $p_n(x) = x_n$ for $x = (x_n)$ and $n \in \mathbb{N}$. Let H be the countable subgroup of $C(X, G)$ generated by all the p_n 's. Let $K \subset C(X, G)$ be the countable subgroup defined as follows: if $f \in C(X, G)$, then $f \in K$ if and only if there exist a finite decomposition $X = Y_1 \cup \dots \cup Y_n$, $Y_i \in \mathcal{B}$, and elements $f_1, \dots, f_n \in H$ such that $f|_{Y_i} = f_i|_{Y_i}$, $i = 1, \dots, n$. Clearly for every $x \in X$ the evaluation map $f \mapsto f(x)$ sends H and K onto G_x . We are going to introduce a metrizable group topology on K such that all the evaluation maps $K \rightarrow G_x$ are open.

Let $\mathcal{N}(G)$ be the filter of neighborhoods of 1_G in G (we'll use a similar notation also for other groups). For $U \in \mathcal{N}(G)$ let $W_U = \{f \in K : f(X) \subset U\}$. The filter \mathcal{F}_0 on K generated by the collection $\{W_U : U \in \mathcal{N}(G)\}$ may be not invariant under inner automorphisms; let \mathcal{F} be the smallest invariant filter containing \mathcal{F}_0 . The filter \mathcal{F} is generated by sets of the form gW_Ug^{-1} ($U \in \mathcal{N}(G)$, $g \in K$). Since K is countable and G is metrizable, \mathcal{F} has a countable base. Equip K with the group topology for which $\mathcal{F} = \mathcal{N}(K)$ is the filter of neighborhoods of 1_K . Then K is metrizable.

Pick $x \in X$. The evaluation map $ev_x : K \rightarrow G_x$ defined by $ev_x(f) = f(x)$ clearly is continuous. We must prove that it is also open. It suffices to check that for every $g_1, \dots, g_n \in K$ and $U \in \mathcal{N}(G)$ there exists $V \in \mathcal{N}(G)$ such that

$$V \cap G_x \subset ev_x \left(\bigcap_{i=1}^n g_i W_U g_i^{-1} \right).$$

Put $a_i = ev_x(g_i) = g_i(x) \in G$. Pick a symmetric open $V \in \mathcal{N}(G)$ such that $a_i^{-1}V^3a_i \subset U$ for $i = 1, \dots, n$. We check that V has the required property.

Let $b \in V \cap G_x$. Pick $f' \in H$ such that $f'(x) = b$. There exists a neighborhood $Y \in \mathcal{B}$ of x such that $f'(Y) \subset V$ and $g_i(Y) \subset Va_i$ ($i = 1, \dots, n$). Let $f \in K$ be such that $f|Y = f'|Y$, $f|X \setminus Y = 1_G$. For every $y \in Y$ and $i = 1, \dots, n$ we have $g_i(y) \in Va_i$, $f(y) \in V$, hence

$$g_i(y)^{-1}f(y)g_i(y) \in a_i^{-1}V^3a_i \subset U,$$

and this is trivially true if $y \in X \setminus Y$. It follows that $g_i^{-1}fg_i \in W_U$ and $f \in \bigcap_{i=1}^n g_iW_Ug_i^{-1}$. Thus $b = f(x) \in ev_x(\bigcap_{i=1}^n g_iW_Ug_i^{-1})$. We have proved that $ev_x : K \rightarrow G_x$ is open.

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